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BASIC NOTIONS OF RELATIVISTIC HYDROMAGNETICS

Paul Reichel

APR 15 1958

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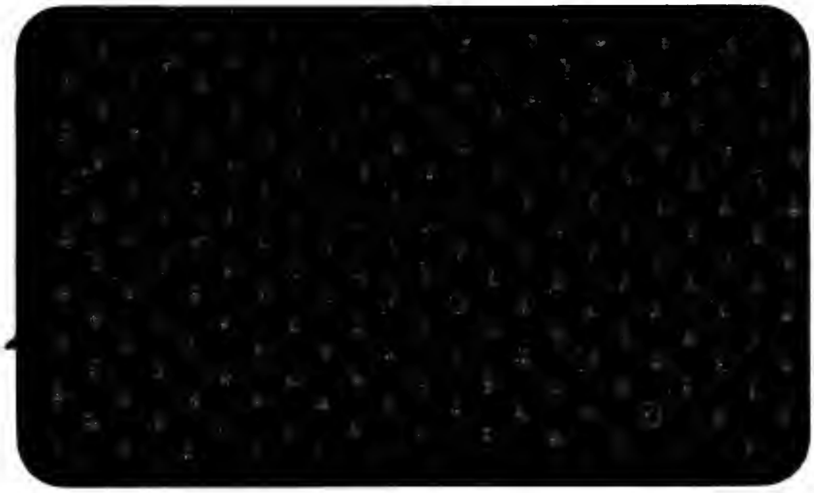

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Contract No. AT(30-1)-1480
With the United States Atomic Energy Commission

NYO-7697
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BASIC NOTIONS OF RELATIVISTIC HYDROMAGNETICS

by

Paul Reichel

ABSTRACT

This report is an elaboration and extension by the author of notes of lectures given by Professor K. O. Friedrichs at New York University in November 1954. The basic notions of relativistic hydromagnetics are presented, and the characteristic speeds of propagation of small disturbances are computed. This report was completed in September 1956.

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BASIC NOTIONS OF RELATIVISTIC HYDROMAGNETICS

Paul Reichel

Part I - Formulation

1. Introduction

The purpose of Part I is to formulate the laws of hydromagnetics in the covariant four dimensional form of special relativity. Taub[2] has formulated the corresponding relativistic hydrodynamic equations for the non-electromagnetic case. On the other hand de Hoffman and Teller[1] have set up shock conservation conditions (jump conditions) for the relativistic treatment of hydromagnetic shocks. These conservation conditions are not adequate to permit of actual solutions for the hydromagnetic field structure, but determine only shock transition values. The present formulation extends the work of de Hoffman and Teller in that it consists of differential equations determining the hydromagnetic field structure (when boundary and initial conditions are given). Thus the conditions of de Hoffman and Teller can be derived from the present equations.

2. Method of Obtaining Formulation - A Search for the Simplest Covariant Expressions

The derivation procedure of a tensor equation (i.e. Lorentz invariant in a (+++-) space) is tantamount to first selecting or constructing "promising" four-dimensional tensors and then examining these to see if the laws of physics may be expressed in terms of equations involving them. An outstanding guiding rule in this procedure (for the case of special relativity only) is: if we can express merely the non-relativistic laws of physics (i.e. equations describing low velocity and low energy effects) in terms of tensors, these equations often constitute automatically the desired invariant forms.

More specifically, tensor equations may be found largely by two techniques:

(a) A classical equation may frequently be enhanced into a tensor equation essentially by "adding an appropriate extra dimension" to the three dimensional "vectors" or matrices appearing in the original equation; and then checking via a Lorentz transformation that such a generalization actually constitutes a tensor equation. This usually means the four dimensional vectors, matrices, etc., entering the equation are individually tensors.

(b) In the procedure of (a), roughly speaking, we selected tensors to fit an equation. Frequently the reverse is done: starting with some tensor an equation is "built around it." It is worth noticing that unlike in the former case, the present tensor need not necessarily have been found by conjecture, but may have been directly constructed via operations on previously

known tensors.

While physical and mathematical considerations are of great value in suggesting possible covariant expressions, it is clear from the foregoing that the actual derivation consists of little more than finally recognizing that a trial expression does in fact transform covariantly, and does have significant physical content. Thus the formulation will merely be stated and examined with no attempt at a systematic derivation.

3. Physical Assumptions (A Relativistic Extension of Lundquist's Equations)

In order to permit reasonable mathematical handling the equations to be formulated will describe an idealization of a somewhat specialized system frequently occurring in nature. The following specific limitations will apply:

(a) The medium in question will consist of several ionic fluid components. We will consider only the case where the relative velocities (diffusion velocities) at any given point between these fluid components is small compared to the speed of light.* (However the relative velocities between different points in the fluid need not be small compared to the speed of light.) This assumption is necessary in order that the system may be described and solved in terms of the total fluid properties without regard to the individual component fluids. Roughly speaking, we may say the system has been reduced to a single effective fluid with a "massless electric current flow tacked on."

(b) We assume the flow to be adiabatic (e.g. zero heat conductivity).

(c) Zero viscosity is assumed.

(d) We assume what is frequently termed "Ohm's law for a perfect conductor": $E + u \times B = 0$. This relation may fail to describe actual dissipationless media in particular when space or time derivatives of the hydromagnetic field variables are too large.

*For example, if in a two-component fluid the components are of say equal mass densities and travel at large equal and opposite velocities through the same region simultaneously, then the velocity of the total fluid is zero and though each fluid component is undergoing a Lorentz contraction, the factor $\sqrt{1-u^2/c^2}$ will be unity. This can be seen to invalidate Equation (6).

Nevertheless it has the advantages of applying in many cases, of simplicity, and of Lorentz invariance of form.

(e) The pressure (relative stress) is assumed to be isotropic.

(f) We assume the dielectric constant and permeability of the medium to approximate that of empty space. (Hence $\mu \chi = c^{-2}$.)

4. Symbols

All definitions refer to the flat space time of special relativity.

The metric tensor:

$$g^{11} = g^{22} = g^{33} = 1$$

$$g^{44} = -c^{-2}$$

$$g_{11} = g_{22} = g_{33} = 1$$

$$g_{44} = -c^{2}$$

All off-diagonal elements are zero.

The 3-space components of fluid velocity: u_x, u_y, u_z.

Conserved rest mass per unit 3-space proper volume $\equiv \rho_o$. The conserved rest mass is defined as the total rest mass less the part we regard as "internal energy." (The meaning of a "true zero level" of internal energy is not important in this connection. It is only necessary that we designate some standard thermodynamic state as corresponding to zero internal energy. The "conserved rest mass" is in fact conserved only if we do not consider creation or annihilation of fundamental particles - e.g. pair production.)

The "four space velocity" vector (as it is often called), K_α or K^α: for $\alpha = 1,2,3$: $K^\alpha = K_\alpha \equiv \dfrac{u_\alpha}{\sqrt{1 - \frac{u^2}{c^2}}}$

$$K^4 \equiv \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$K_4 \equiv - \frac{c^2}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Proper internal energy per unit conserved rest mass $\equiv e_o$.

Scalar invariant pressure $\equiv p$.

Proper enthalpy per unit conserved rest mass $\equiv i_o \equiv e_o + \frac{p}{\rho_o}$.

Electromagnetic tensor: $B_{\alpha\beta} = -B_{\beta\alpha}$. This may be written as the following array:

$$\begin{pmatrix} 0 & B_z & -B_y & E_x \\ -B_z & 0 & B_x & E_y \\ B_y & -B_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{pmatrix}$$

If this is changed to contravariant form the result is as the above array except that the elements forming the fourth row and column are each multiplied by $-c^{-2}$. ($B^{\sigma 4} = -c^{-2} B_{\sigma 4}$ and $B^{4\nu} = -c^{-2} B_{4\nu}$.)

Charge density (seen in any given Lorentz frame) $\equiv q$.

Current four vector: I^α. Fourth contravariant component = q/c^2.

Permeability $\equiv \mu$ and dielectric constant $\equiv \chi$.

Coordinate derivative operator $\equiv \nabla_\alpha \equiv \frac{\partial}{\partial x_\alpha}$.

5. Covariant Formulation

(1) $$K^{\alpha}K_{\alpha} = -c^2 .$$

This takes the form of an identity if we regard the K's as defined in terms of u_x, u_y, u_z:

$$\left(\frac{u_x}{\sqrt{1-\frac{u^2}{c^2}}}\right)^2 + \left(\frac{u_y}{\sqrt{1-\frac{u^2}{c^2}}}\right)^2 + \left(\frac{u_z}{\sqrt{1-\frac{u^2}{c^2}}}\right)^2 - \left(\frac{c}{\sqrt{1-\frac{u^2}{c^2}}}\right)^2 \equiv -c^2$$

Thus, in the three dimensional formulation where the u_x, u_y, u_z variables are included, (1) will be an identity; while in the equivalent four dimensional formulation where these are not included, (1) will be a constraint.

(2) (REFERS TO A <u>GIVEN</u> ELEMENT OF FLUID): $p = p(\rho_o)$

This general expression for the equation of state indicates that only one thermodynamic variable is needed to determine the state, in accordance with our assumption that the flow is adiabatic. While this form of equation applies to each fluid element, we have not assumed that all elements obey the same particular equation of state. (i.e. The formulation is not restricted to a system in which the entropy per unit conserved rest mass is space independent.)

(3) $$B_{\alpha\beta}K^{\beta} = 0 .$$

This relation is the covariant form of what is known as "Ohm's law for a perfect conductor", and states that an observer

stationary with respect to the fluid sees zero electric field. Equation (3) is completely equivalent to the following three dimensional expression:

(3A) $$E + u \times B = 0$$

(3B) $$E \cdot u = 0 .$$

Thus (3B) merely states that E is perpendicular to u, a fact already obvious from (3A).

If (3) be premultiplied by K^{α}, then because of the skew symmetry of $B_{\alpha\beta}$, the result is identically zero:

(3C) $$K^{\alpha}B_{\alpha\beta}K^{\beta} \equiv 0$$

or $$K^{1}(B_{1\beta}K^{\beta}) + K^{2}(B_{2\beta}K^{\beta}) + K^{3}(B_{3\beta}K^{\beta}) + K^{4}(B_{4\beta}K^{\beta}) \equiv 0.$$

Since the four brackets are respectively the four expressions represented by (3) we see that any one of these four expressions is linearly dependent on the other three. This then constitutes an alternate proof that (3B) is merely a consequence of (3A).

(4) $$\nabla_{\alpha}B_{\beta\gamma} + \nabla_{\beta}B_{\gamma\alpha} + \nabla_{\gamma}B_{\alpha\beta} = 0.$$

This is a tensor equation, for cyclic differentiation of a skew symmetric tensor yields a skew symmetric tensor one rank higher (in the flat space time of special relativity). Equation (4) is equivalent to two of the Maxwell equations:

(4A) $$\mathrm{curl}\ E + \frac{\partial B}{\partial t} = 0$$

(4B) $$\mathrm{div}\ B = 0.$$

Equation (4B) is linearly independent of the three equations

comprising (4A), but in a rather trivial way. Thus (4A) implies not (4B), but the weaker statement that $\frac{\partial}{\partial t}$ div B = 0. However, if we assume as a supplementary condition that at any one time div B = 0 at a point then it will always be so at that point. that is (4B) follows from (4A) combined with the physically reasonable assumption that div B vanished at some time or other in each part of the region of interest.

The dependence between the "weakened" (4B), namely that $\frac{\partial}{\partial t}$ div B = 0, and (4A) may also be seen by cyclically differentiating (4):

$$\begin{aligned} &\nabla_\alpha(\nabla_\beta B_{\gamma\delta}+\nabla_\gamma B_{\delta\beta}+\nabla_\delta B_{\beta\gamma})+\nabla_\beta(\nabla_\alpha B_{\gamma\delta}+\nabla_\gamma B_{\delta\alpha}+\\ (4C)\qquad &\nabla_\delta B_{\alpha\gamma})+\nabla_\gamma(\nabla_\alpha B_{\beta\delta}+\nabla_\beta B_{\delta\alpha}+\nabla_\delta B_{\alpha\beta})+\\ &\nabla_\delta(\nabla_\alpha B_{\beta\gamma}+\nabla_\beta B_{\gamma\alpha}+\nabla_\gamma B_{\alpha\beta})\equiv 0\ . \end{aligned}$$

This is seen to be an identity because of the skew symmetry of $B_{\alpha\beta}$. Taking $\delta = 4$, the fourth term becomes $\frac{\partial}{\partial t}$ div B, which is seen to be linearly dependent on space derivatives of the three equations comprising (4A).

$$(5)\qquad \mu^{-1}\ \nabla_\beta B^{\alpha\beta} = I^\alpha\ .$$

This is equivalent to two of the Maxwell equations:

$$(5A)\qquad \text{curl } H = \frac{\partial D}{\partial t} + I$$

$$(5B)\qquad \text{div } D = q \quad (\equiv c^2 I^4)\ .$$

Equation (5B) and the three equations comprising (5A) are linearly independent. Equation (5) will not enter explicitly

into the formulation, but will be eliminated by replacing I^α wherever it appears by $\mu^{-1} \nabla_\beta B^{\alpha\beta}$. It is however interesting to examine (5) briefly:

For the I^α to form a non-zero four vector it is readily seen that we cannot have $q \doteq 0$ in all reference frames. In other words, except for the case of $I \equiv 0$, it is impossible for a fluid to be neutral in all reference frames. To understand this physically, we may consider a two component fluid in which the positive part is moving at some given velocity with respect to the negative part. Then an observer stationary with respect to the negative part sees the positive part moving past him and therefore of increased charge density due to a Lorentz contraction. On the other hand an observer stationary with respect to the positive charge does not see it under a Lorentz contraction, but instead sees the negative charge "compressed". The fact that different observers may disagree on the relative number of positive and negative particles in a given volume element of the medium does not constitute an inconsistency, but is in fact a consequence of the different meanings of simultaneity for the different observers. Clearly even if the diffusion velocity is small compared to that of light, a medium may display opposite net charge signs in two slightly different Lorentz frames.

Equation (5) (or what is the same thing, (5A) and (5B)) implies the conservation of charge; for by taking the divergence of (5A) and the time derivative of (5B), the E variable may be eliminated, yielding the relation:

$$\frac{\partial q}{\partial t} + \frac{cI_x}{\partial x} + \frac{\partial I_y}{\partial y} + \frac{\partial I_z}{\partial z} = 0 .$$

(6) $$\nabla_\alpha(\rho_o K^\alpha) = 0 .$$

Equation (6) states the conservation of the quantity we have accordingly called "the conserved rest mass." This relation has been used by Taub in the corresponding non-electromagnetic formulation (reference [2]). In terms of the fluid velocity u it becomes:

$$\nabla_t \frac{\rho_o}{\sqrt{1 - \frac{u^2}{c^2}}} + \nabla \cdot \frac{\rho_o u}{\sqrt{1 - \frac{u^2}{c^2}}} = 0.$$

Remembering that ρ_o is <u>not</u> the conserved rest mass per unit volume seen by an observer whom the fluid is passing at speed u, but is defined as the conserved rest mass density in the proper frame (i.e. stationary with respect to the fluid), we recognize the necessity of the factors $(1 - \frac{u^2}{c^2})^{-1/2}$ in (6A). This conservation law must be qualified as follows:

(a) There are notable exceptions to this conservation law. For example if an electron and positron annihilate each other by forming electromagnetic radiation, the original rest mass simply disappears, since photons have zero rest mass. Similar exceptions are to be found in the various nuclear reactions, etc. There are actually two conservation laws instead of one (to the same approximation): energy and the number of particles.

(b) In order that a conservation law apply, the quantity to be designated as "conserved rest mass" must <u>by definition</u> be set aside from the internal energy. Thus electromagnetic radiation

which has zero rest mass, can supply internal energy to a material (by absorption or adiabatically by say a "radiation pressure compression"); so that including the internal energy in the definition of conserved rest mass would in fact violate the desired conservation property. It need not be supposed however that there is some inherent "true zero level" of internal energy, so that the total proper mass may be uniquely subdivided into a conserved rest mass part and an internal energy part. It is sufficient merely to regard the conserved rest mass as the total proper mass of some arbitrarily chosen "standard state", and then to interpret the internal energy of any other state as either positive or negative depending upon whether the total proper mass is respectively more or less than in the standard state. (Negative internal energy is a reasonable concept: For example at 0^{o} Kelvin the negative lattice binding energy of a material far outweighs the slight positive zero-point vibrational energy.)

Since (6) permits of the above very arbitrary definition of conserved rest mass - as some constant part of the variable total proper mass - it would appear that the equation is too "artificial" to be of much virtue. In other words it may be asked why do we not simply reduce the number of variables and equations by one by removing the conserved rest mass and enthalpy variables and attempting to employ instead a single total proper mass variable, and dropping (6)? Actually the importance of the concept of conserved rest mass arises because in relativistic hydrodynamics it appears to be perhaps the unique quantity through which we may

quantitatively designate a discrete element of fluid (unless we borrow microscopic concepts to designate it - e.g. ions entering and leaving the element). Thus a discrete element of fluid is defined as the fluid within a closed surface which moves so that there is no net flow of conserved rest mass* across it at any point. From this definition (and (6)) we see that the conserved rest mass of an element of fluid is a constant of the motion. Hence the mathematical usefulness of the concept of conserved rest mass arises because it enables us to use a convenient type of Lagrangian derivative formulation; for without the conserved rest mass quantity a Lagrangian formulation would require "semi-quantitative" variables like $\left(\frac{\text{ENTHALPY}}{\text{GIVEN FLUID ELEMENT}}\right)$, but with the help of the conserved rest mass quantity this variable assumes the quantitatively meaningful form $\left(\frac{\text{ENTHALPY}}{\text{UNIT CONSERVED REST MASS}}\right)$.

(7) $$\rho_o K^\beta \nabla_\beta (1+c^{-2} i_o) K_\alpha + \nabla_\alpha p + \mu^{-1} B_{\alpha\gamma} \nabla_\beta B^{\beta\gamma} = 0 \ .$$

This is equivalent to the following two equations (using the fact that we have assumed $\mu \varkappa = c^{-2}$):

(7A) $$\frac{\rho_o}{\sqrt{1-\frac{u^2}{c^2}}} \frac{d}{dt}\left[(1+\frac{i_o}{c^2})\frac{u}{\sqrt{1-\frac{u^2}{c^2}}}\right] + \nabla p + B x \left\{\nabla \, x H - \nabla_t D\right\} - E(\nabla \circ D) = 0$$

(7B) $$\frac{-\rho_o c^2}{\sqrt{1-\frac{u^2}{c^2}}} \frac{d}{dt}\left[(1+\frac{i_o}{c^2})\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\right] + \frac{\partial p}{\partial t} + E \circ \left\{\nabla x H - \nabla_t D\right\} = 0.$$

---

*Microscopically this means no net flow of particles.

Eouation (7A) is the equation of motion corresponding to Newton's second law. Eouation (7B) describes (but not very obviously) the rate of change of the total (non-electromagnetic) mass energy of an element of fluid due to a combination of mechanical and electromagnetic effects. In the proper frame we have from (3) that E = 0, so that (7B) reduces in that frame to

$$-\rho_o \frac{d}{dt}(i_o) + \frac{dp}{dt} = 0 ,$$

or

$$\text{(7C)} \qquad di_o = \frac{dp}{\rho_o} .$$

This relation, which is a consequence of (3) and (7B), states that the entropy of a given element of fluid (appropriately defined) remains unchanged. (Using the definition $i_o \equiv e_o + p\rho_o^{-1}$, (7C) can readily be shown to describe an element of fluid when the only internal energy changes are due to compression - not heat conduction or diffusion.) This will render the equation of state an "adiabatic" one.

Equation (7B) is not a consequence of (7A) alone, but will now be shown to be a consequence of (1), (3), (7A) and (7C). If (7) is premultiplied by K^α we get a linear combination of the four equations represented by (7). Performing the indicated differentiations and putting $di_o = \rho_o^{-1}dp_o$, this linear combination is

$$\rho_o(1+c^{-2}i_o)K^\beta K^\alpha \nabla_\beta K_\alpha + K^\alpha K^\beta K_\alpha c^{-2} \nabla_\beta p + K^\alpha \nabla_\alpha p = 0.$$

The electromagnetic term has been dropped, since it is seen to equal zero by virtue of (3). Next we observe that

$$K^\alpha \nabla_\beta K_\alpha \equiv K_\alpha \nabla_\beta K^\alpha \ .$$

Hence the preceding expression may be written

$$\tfrac{1}{2}\rho_o(1+c^{-2}i_o)K^\beta(K^\alpha \nabla_\beta K_\alpha + K_\alpha \nabla_\beta K^\alpha) + K^\alpha K^\beta K_\alpha c^{-2} \nabla_\beta p + K^\beta \nabla_\beta p = 0$$

or

$$\tfrac{1}{2}\rho(1+c^{-2}i_o)K^\beta \nabla_\beta(K^\alpha K_\alpha + c^2) + c^{-2}K^\beta \nabla_\beta p(K^\alpha K_\alpha + c^2) = 0 \ .$$

If, in addition to having assumed (3), we now also use (1), it follows from this last expression that the four equations represented by (7) are linearly dependent. Hence (7B) is a consequence of (7A), (7C), (1), and (3).

The term in (7A) which contains the Lagrangian time derivative is identified as the time rate of change of momentum per unit volume of the fluid element (i.e. momentum and volume seen by an observer moving at velocity - $\vec{u}$ with respect to the fluid). It is interesting to examine (7A) for the case $u \ll c$. The momentum per unit volume is seen to become (remembering the definition $i_o \equiv e_o + \frac{p}{\rho_o}$):

$$\rho_o(1 + \frac{e_o}{c^2} + \frac{p}{\rho_o}) \ u.$$

Since the total mass density is $\rho_o(1+ \frac{e_o}{c^2})$, the inclusion of the p/ρ_o term in the above expression might appear strange. The physical reason that p/ρ_o actually does contribute to the inertia arises from the fact that there is a supplementary flow of energy across a fluid element due to the work done by the pressure of the element on its surroundings as the element moves. (That is

the neighboring fluid does work on the receding side of the element while it receives work from the advancing side, so that energy must be flowing across the element.) This flowing energy has inertia which resists a change of flow rate. An interesting consequence of the fact that the enthalpy - and not merely the internal energy - contributes to the inertia is that for sufficiently high pressures a relativistic increase of inertia occurs even when $u/c \to 0$.

6. Determinacy of System

We wish to compare the number of independent equations and variables to indicate the determinacy of the system. (B and H will not be counted as different variables, as we will implicitly assume the constitutive relation $B = \mu H$. Similarly we implicitly assume $D = \chi E$.):

A. Four dimensional formulation

(1) Consists of 1 equation and 4 unknowns: K_1, K_2, K_3, K_4.

(2) Consists of 1 equation and 2 unknowns: p, ρ_o.

(3) Consists of 3 independent equations and adds 6 new unknowns to those already appearing in (1) and (2): $E_1, E_2, E_3, B_1, B_2, B_3$.

(4) Consists of 3 independent equations and adds no new unknowns.

(6) Consists of 1 equation and adds no new unknowns.

(7) Consists of 4 independent equations and adds 1 new unknown: i_o.

Thus we have 13 equations in 13 unknowns, so that the system is determined (when boundary and initial conditions are prescribed).

B. Three dimensional formulation

(2) Consists of 1 equation and 2 unknowns: p, ρ_o.

(3A) Consists of 3 independent equations and 9 unknowns: $E_1, E_2, E_3, B_1, B_2, B_3, u_1, u_2, u_3$.

(4A) Consists of 3 independent equations and adds no new unknowns.

(6A) Consists of 1 equation and adds no new unknowns.

(7A) Consists of 3 equations and adds 1 new unknown: i_o.

(7B) Consists of 1 equation and adds no new unknowns.

Again this system is determined since we have 12 equations in 12 unknowns.

7. The Stress Energy Tensor (Alternate Formulation)

The following definition constitutes a combination of the mechanical and electromagnetic stress energy tensors:

$$(8) \quad T^{\beta}_{\alpha} \equiv \rho(1+c^{-2}i_o)K_\alpha K^\beta + p\delta^\beta_\alpha - \mu^{-1}B_{\alpha\gamma}B^{\gamma\beta} + (4\mu)^{-1}B^{\gamma\delta}B_{\gamma\delta}\delta^\beta_\alpha \ .$$

It is obvious that $T_{\alpha\beta} = T_{\beta\alpha}$.

A generalization of (7) in that it includes also electromagnetic mass and momentum, as well as mechanical, can be written in terms of this tensor in the form:

$$(7') \qquad \nabla_\beta T^\beta_\alpha = 0.$$

Equation (7'), while in itself not equivalent to (7), could nevertheless be used in its place without altering the content of the original system of equations. (7') is conservation form, (7) is not. The reason the formulation would be equivalent lies in that (7) and (7') differ essentially only in that the latter contains additional information derived from Maxwell's equations appearing separately in the formulations anyway. Equation (7') is equivalent to the following two equations:

$$(7'A) \quad \nabla_t \left\{ \frac{\rho_o(c^2+e_o)+p}{c^2-u^2}\, u + \frac{ExB}{\mu c^2} \right\} + \nabla \cdot \left\{ \frac{\rho_o(c^2+e_o)+p}{c^2-u^2}\, u + \frac{BB}{\mu} + \frac{EE}{\mu c^2} \right\}$$
$$+ \nabla \left\{ \rho_o + \frac{B^2}{2\mu} + \frac{E^2}{2\mu c^2} \right\} = 0$$

$$(7'B) \quad \nabla_t \left\{ \frac{\rho_o(c^2+e_o)+p\,\frac{u^2}{c^2}}{c^2-u^2} + \frac{B^2}{2\mu} + \frac{E^2}{2\mu c^2} \right\} + \nabla \circ \left\{ u\, \frac{\rho(c^2+e_o)+p}{1-\frac{u^2}{c^2}} + \frac{ExB}{\mu} \right\} =$$

Equation (7'A) describes the time rate of change of the combined mechanical and electromagnetic momentum at a point. Equation (7'B) is the relativistic form of the law of conservation of energy. Clearly the quantity

$$\frac{\rho_o(c^2+e_o) + p\,\frac{u^2}{c^2}}{1-\frac{u^2}{c^2}} + \frac{B^2}{2\mu} + \frac{E^2}{2\mu c^2}$$

is to be identified with the total electro-mechanical mass energy density as seen by an observer moving at velocity -u with respect to the proper frame. Also the net energy flow density seen by this observer is obviously:

$$u\,\frac{\rho_o(c^2+e_o)+p}{1-\frac{u^2}{c^2}} + \frac{E\text{x}B}{\mu} \;.$$

The electromagnetic parts of energy and power density are seen to be identical with the classical form, while the mechanical parts differ markedly. In particular it may be asked why the mechanical energy includes the $p\,u^2c^{-2}$ term in the numerator. The reason is essentially that an observer in going from rest to some finite velocity with respect to a fluid element sees it undergo a Lorentz contraction while under the pressure of the surrounding medium, which, to him constitutes work being done on the element: thus the internal energy density seen by an observer moving with respect to the fluid is increased more than merely by the factor $(1-u^2c^{-2})^{-1}$ when pressure is involved. If $u \ll c$ and (6A) is subtracted from (7'B) the classical form of energy conservation is obtained.

It is worth remarking that the above identification of total energy and power densities as seen by an observer in relative motion, while supplementary to the already determinate formulation, is in an important sense, not trivial. For the quantities ρ_o, e_o, and i_o ($\equiv e_o + p\rho_o^{-1}$) entering the formulation in no way serve to disclose the corresponding densities of matter, internal energy, or enthalpy as seen by an observer in relative motion. Thus if the formulation is to be related to quantities measurable by an observer in an arbitrary velocity frame these supplementary identifications must be assumed.

Part II - Small Amplitude Hydromagnetic Disturbance Velocities

1. Introduction

The system of equations formulated in Part I will be shown to be hyperbolic. The characteristic velocities of small amplitude disturbances will be obtained. It will be seen that unless restrictions are placed on the mechanical parameters describing the medium, disturbances propagating faster than the speed of light would be possible.

Computations of relativistic hydromagnetic shock (i.e. finite amplitude) propagation velocities have been made for certain particular cases of relative orientation between the vectors involved (e.g. for the magnetic field parallel to the shock front; or for the material velocity jump vector parallel to the shock front normal, etc.) by de Hoffmann and Teller[1]. In the present case, while treating only small amplitudes, we will consider characteristic velocities of disturbances of all permissible orientations between the field vectors.

2. Equations and Assumptions

The non-relativistic counterpart of the results to be obtained here has appeared in Reference [7] (i.e. for the standard Lundquist equations). Since it will be interesting to compare the present results with those of Reference [7], it is convenient to employ the three-dimensional relativistic formulation of Part I.

The three dimensional relativistic formulation consists of Equations (2), (3A), (4A), (6A), (7A), (7B). In addition to these equations we assume the following: only small amplitude waves are to be considered. We consider only the proper frame of the fluid. (If a disturbance advances more slowly than light in the proper frame, it will also be slower in all other Lorentz frames.) The charge density in the proper frame is assumed to be zero.

3. <u>Equation for Characteristic Velocities</u>

Using (3A) we eliminate the variable E from the other equations of the formulation. Within the context of obtaining characteristic velocities*, we may conclude that Equation (2) implies the relationship

$$\frac{\partial p}{\partial x_\alpha} = \frac{dp}{d\rho_o} \frac{\partial \rho_o}{\partial x_\alpha} .$$

Using this ∇p is replaced by $\frac{dp}{d\rho_o} \nabla \rho_o$ in (7A). Then the remaining equations of our formulation become:

$$(4A_1) \qquad \nabla_t H + \nabla x (H \times u) = 0$$

$$(6A_1) \qquad \nabla_t \left(\frac{\rho_o}{\sqrt{1 - \frac{u^2}{c^2}}} \right) + \nabla \cdot \left(\frac{\rho_o u}{\sqrt{1 - \frac{u^2}{c^2}}} \right) = 0$$

$$(7A_1) \qquad \frac{\rho_o}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{d}{dt} \left(\frac{u(1 + \frac{i_o}{c^2})}{\sqrt{1 - \frac{u^2}{c^2}}} \right) + \frac{dp}{d\rho_o} \nabla \rho_o + \mu H x \left\{ \nabla x H - \nabla_t \chi \mu H x u \right\} -$$

$$- (\mu H x u)(\nabla \cdot \chi \mu H x u) = 0$$

$$(7B_1) \qquad \frac{-\rho_o c^2}{\sqrt{1 - \frac{u^2}{c^2}}} \frac{d}{dt} \left((1 + \frac{i_o}{c^2}) \frac{1}{\sqrt{1 + \frac{u^2}{c^2}}} \right) + \frac{\partial p}{\partial t} + (\mu H x u) \cdot \left\{ \nabla x H - \nabla_t \chi \mu H x u \right\} = 0.$$

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*The characteristic velocity formalism to be employed is in effect as though the discontinuities correspond simply to the limiting case of very large space and time derivatives. Thus, within such a local x,y,z,t region of sharp (but slight) change of the state variables, the "operator of the equation of state" (equation 2) is comparatively space and time independent.

In the proper frame $(u = 0)$ $(7B_1)$ becomes, assuming small amplitude disturbances:

$$(7B_2) \qquad \frac{\partial i_o}{\partial t} = \frac{1}{\rho_o} \frac{\partial p}{\partial t} .$$

This relation does not contribute to the set of simultaneous equations needed to determine the characteristic velocities (since a linear combination of <u>both</u> time and space derivatives is required). Further since we wish to carry the i_o variable along in the formulation, $(7B_2)$ (which, in conjunction with (2), states that i_o is a function of ρ_o only) will not be used to eliminate i_o. Thus we do not use $(7B_2)$. Assuming small amplitudes and the proper frame for Equations $(4A_1)$, $(6A_1)$, $(7A_1)$, the formalism of the theory of characteristics* leads to the relations

$$(4A_2) \qquad -v\delta H + H\delta u_n - H_n \delta u = 0$$

$$(6A_2) \qquad -v\delta\rho_o + \rho_o \delta u_n = 0$$

$$(7A_2) \quad -\rho_o\left(1+ \frac{i_o}{c^2}\right)v\delta u + a^2 n\delta\rho_o + \mu n(H\cdot\delta H) - \mu H_n \delta H + \chi\mu^2 vH(H\cdot\delta u) - \chi\mu^2 vH^2\delta u = 0$$

where $v \equiv$ characteristic velocity (a number)

$n \equiv$ unit normal vector to wave front

$H_n \equiv n\cdot H$

$u_n \equiv n\cdot u$

$a^2 \equiv \dfrac{dp}{d\rho_o}$

$\delta(\ )$ indicates jump (small in this case) in a variable.

Equations $(4A_2)$, $(6A_2)$, $(7A_2)$ parallel closely their non-relativistic counterpart presented on page 10 of Reference [7].

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*See for example Reference [7].

In fact $(4A_2)$ and $(6A_2)$ are seen to be identical respectively to (B_1) and (B_3) of Reference [7]; while $(7A_2)$ differs from (B_2) by a factor in the first term, and also in that the last two terms in $(7A_2)$ do not appear in (B_2). The derivation of these two additional terms is given in Appendix 1.

Substituting into $(7A_2)$ for δH (via $(4A_2)$) and for $\delta\rho_o$ (via $(6A_2)$) gives

$$(7A_3) \quad v^2 \left\{[\rho_o(1+ \frac{i_o}{c^2})+\chi\mu^2 H^2]\delta\vec{u}-\vec{H}[\chi\mu^2(\vec{H}\cdot\delta\vec{u})]\right\}$$
$$= \rho_o a^2\vec{n}\delta u_n+\mu\vec{n}(\vec{H}\cdot\left\{\vec{H}\delta u_n-H_n\delta\vec{u}\right\})-\mu H_n(\vec{H}\delta u_n-H_n\delta\vec{u}) \ .$$

Equation $(7A_3)$ constitutes 3 homogeneous equations in the variables δu_x, δu_y, δu_z. The determinant of this set of equations is*

$$\begin{vmatrix} -\rho_o v^2(1+\frac{i_o}{c^2})+\rho_o a^2\frac{H_n^2}{H^2} & \rho_o a^2\frac{H_n}{H^2}\sqrt{H^2-H_n^2} & 0 \\ \rho_o a^2\frac{H_n}{H^2}\sqrt{H^2-H_n^2} & -\rho_o v^2(1+\frac{i_o}{c^2}+\frac{\chi\mu^2H^2}{\rho_o})+\rho_o a^2(\frac{H^2-H_n^2}{H^2})+\mu H^2 & 0 \\ 0 & 0 & -\rho_o v^2(1+\frac{i_o}{c^2}+\frac{\chi\mu^2H^2}{\rho_o})+\mu H_n^2 \end{vmatrix}$$

This is a polynomial cubic in (v^2). (The determinant of equations $(7A_3)$ should be essentially the same as the determinant for the 7 equations constituting $(4A_2)$, $(6A_2)$, $(7A_2)$ - except for the possibility of zero roots (contact discontinuities). This possibility is readily examined by putting $v = 0$ into $(4A_2)$, $(6A_2)$, $(7A_2)$, and observing that the resulting relations are in fact permissible for appropriate parameter values.) Writing it out

*For a derivation of this determinant, see Appendix 2.

and equating to zero gives the equation for the 3 values of v^2.

$$(7A_4) \quad 0=\rho_o \left\{-v^2(\rho_o[1+\frac{i_o}{c^2}]+\chi\mu^2H^2)+\mu H_n^2\right\}\left\{v^4\left[(\rho_o[1+\frac{i_o}{c^2}]+\chi\mu^2H^2)[1+\frac{i_o}{c^2}]\right]\right.$$

$$\left. - v^2\left[(\rho_o a^2+\mu H^2)[1+\frac{i_o}{c^2}]+ a^2H_n^2\chi\mu^2\right]+ a^2\mu H_n^2\right\}$$

If we set equal to zero the relativistically introduced quantities $\frac{i_o}{c^2}$ and $\chi\mu^2H^2$ (due to displacement current), $(7A_4)$ reduces to (2.7) on page 11 of Reference [7]. (Except that the latter has an additional root of $v^2 = 0$.)

4. Examination of Characteristic Velocities

A. The system is hyperbolic

The last factor of the determinant $(7A_4)$ is quadratic in (v^2) and has the discriminant

$$b^2-4ac \equiv \left[a^2(\rho_o[1+\frac{i_o}{c^2}] + H_n^2\varkappa\mu^2)\right]^2 + \left[\mu H^2[1+\frac{i_o}{c^2}]\right]^2$$

$$+2\left[a^2(\rho_o[1+\frac{i_o}{c^2}]+H_n^2\varkappa\mu^2)\right]\left[\mu H^2[1+\frac{i_o}{c^2}]\right]$$

$$-4(\rho_o[1+\frac{i_o}{c^2}] + H^2\varkappa\mu^2)[1+\frac{i_o}{c^2}](a^2\mu H_n^2).$$

It is readily seen that the discriminant cannot be negative. It is thus further seen that both solutions for v^2 obtained from the last factor of the determinant are positive real. Clearly all 6 characteristic velocities obtained from $(7A_4)$ are real.

B. Intermediate wave cannot exceed speed of light

The factor linear in (v^2) of (2D) gives what corresponds to the "intermediate wave" of Reference [7]:

$$(7A_5) \qquad v^2_{INT.} = \frac{\mu H_n^2}{\rho_o[1+\frac{i_o}{c^2}]+\varkappa\mu^2H^2}$$

Because of the additional terms in the denominator in the present case (as compared to Reference [7], we see that the intermediate wave cannot exceed the speed of light. (Speed of light $= (\varkappa\mu)^{-1/2} = c$, in the present formulation.)

C. Fast wave not identically slower than light, but slow wave is

To examine more concisely the two solutions for (v^2) obtained from the last factor of the determinant we introduce the notation

$$R \equiv \rho_o + \frac{\chi\mu^2 H^2}{1+\frac{i_o}{c^2}}, \qquad r = \rho_o + \frac{\chi\mu^2 H_n^2}{1+\frac{i_o}{c^2}}, \qquad \therefore \rho_o \leq r \leq R.$$

Then the solutions for v^2_{FAST} and v^2_{SLOW} are

$$(7A_6) \qquad v^2_{\substack{FAST\\SLOW}} = \frac{a^2r+\mu H^2 \pm \sqrt{(a^2r+\mu H^2)^2 - 4Ra^2\mu H_n^2}}{2R(1+\frac{i_o}{c^2})}$$

From $(7A_6)$ it is seen that the slow wave velocity cannot exceed the speed of light. (To see this compare v^2_{SLOW} with the corresponding expression obtained if the term $-4Ra^2\mu H_n^2$ in the radicand is replaced by the term of larger magnitude $-4ra^2\mu H^2$.)

The fast wave velocity is not greater than the speed of light unless $a(1+i_o c^{-2})^{-1/2}$ is greater than the speed of light. To show this we solve $(7A_6)$ (upper sign option) for a^2. The result is:

$$a^2 = v^2_{FAST}(1+\frac{i_o}{c^2}) \frac{v^2_{FAST}\rho_o(1+\frac{i_o}{c^2}) + H(v^2_{FAST}\chi\mu^2-\mu)}{v^2_{FAST}\rho_o(1+\frac{i_o}{c^2})+H_n^2(v^2_{FAST}\chi\mu^2-H)}$$

This expression immediately indicates that if $v_{FAST} > \frac{1}{\sqrt{\mu\chi}}$ then $\frac{a}{\sqrt{1+\frac{i_o}{c^2}}} > \frac{1}{\sqrt{\mu\chi}}$.

Special Cases of $(7A_6)$

Some of these will illustrate that the speed of light is exceeded if a^2 is sufficiently large:

If $H_n = 0$

$$v^2_{FAST} = \frac{a^2 + \frac{\mu H^2}{\rho_o}}{1+\frac{i_o}{c^2}+\frac{\chi\mu^2H^2}{\rho_o}} \qquad v^2_{SLOW} = 0$$

If $H_n = |H|$

$$v^2_{\substack{\text{SLOW} \\ \text{(OR FAST)}}} = \frac{a^2}{1 + \frac{i_o}{c^2}} \qquad v^2_{\substack{\text{FAST} \\ \text{(OR SLOW)}}} = \frac{\frac{\mu H^2}{\rho_o}}{1 + \frac{i_o}{c^2} + \frac{\chi\mu^2 H^2}{\rho_o}}$$

The choice of upper or lower index in the last two expressions (indices coupled) depends upon the comparative numerical values of the respective expressions. The equation of state of the gas must satisfy $\sqrt{\frac{dp}{d\rho}} = a < c\sqrt{1 + i_o/c^2}$. In the last expression (Alfvén wave case) $H_n = |H|$ to first order only.

APPENDIX I

Evaluation of Expression Corresponding to $B \times \frac{\partial}{\partial t}(B \times u)$ for Characteristic Velocity Formalism

$$B \times \frac{\partial}{\partial t}(B \times u) = \hat{i}\left(B_y \frac{\partial}{\partial t}(B \times u)_z - B_z \frac{\partial}{\partial t}(B \times u)_y\right)$$
$$+ \hat{j}(\qquad\qquad)$$
$$+ \hat{k}(\qquad\qquad)$$

$$= \hat{i}(B_y \frac{\partial}{\partial t}(B_x u_y - u_x B_y) - B_z \frac{\partial}{\partial t}(u_x B_z - B_x u_z))$$
$$+ \hat{j}(\qquad\qquad)$$
$$+ \hat{k}(\qquad\qquad)$$

Assuming small amplitudes we have $\frac{\partial}{\partial t} B_i u_j = B_i \frac{\partial u_j}{\partial t}$. That is the B components here act as constants with respect to the time derivative operator. Using this fact, we get after some manipulation:

$$B \times \frac{\partial}{\partial t}(B \times u) = \hat{i}(B_y B_x \frac{\partial u_y}{\partial t} + B_x B_z \frac{\partial u_z}{\partial t} - B^2 \frac{\partial u_x}{\partial t} + B_x^2 \frac{\partial u_x}{\partial t})$$
$$+ \hat{j}(\qquad\qquad)$$
$$+ \hat{k}(\qquad\qquad)$$

$$B \times \frac{\partial}{\partial t}(B \times u) = \hat{i}(B_x(B \cdot \frac{\partial u}{\partial t}) - \frac{\partial u_x}{\partial t} B^2)$$
$$+ \hat{j}(B_y(B \cdot \frac{\partial u}{\partial t}) - \frac{\partial u_y}{\partial t} B^2)$$
$$+ \hat{k}(B_z(B \cdot \frac{\partial u}{\partial t}) - \frac{\partial u_z}{\partial t} B^2)$$

$$B \times \frac{\partial}{\partial t}(B \times u) = B(B \cdot \frac{\partial u}{\partial t}) - B^2 \frac{\partial u}{\partial t} .$$

Under the formalism for characteristics we replace $\frac{\partial u}{\partial t}$ by $-v\delta u$. Hence $B \times \frac{\partial}{\partial t}(B \times u)$ corresponds to: $-vB(B \cdot \delta u) + vB^2 \delta u$.

APPENDIX II

Obtaining the Determinant of the Three Equation System Constituting the Relation ($7A_3$)

Choosing the field H to be in the x-direction (to first order) the three equations comprising the relation ($7A_3$) may be written (after dropping higher order terms):

x component
$$\rho_o v^2\left(1+\frac{i_o}{c^2}\right)\delta u_x = \rho_o \delta u_x\left\{a^2 n_x^2\right\}+\rho_o\delta u_y\left\{a^2 n_x n_y\right\}+ + \rho_o\delta u_z\left\{a^2 n_x n_z\right\}$$

y component
$$\rho_o v^2\left(1+\frac{i_o}{c^2}+\frac{\chi\mu^2H^2}{\rho_o}\right)\delta u_y = \rho_o\delta u_x\left\{a^2 n_y n_x\right\}+\rho_o\delta u_y\left\{a^2 n_y^2 + \frac{\mu H_x^2}{\rho_o}(n_x^2+n_y^2)\right\} +\rho_o\delta u_z\left\{a^2 n_y n_z+\frac{\mu H_x^2}{\rho_o}n_y n_z\right\}$$

z component
$$\rho_o v^2\left(1+\frac{i_o}{c^2}+\frac{\chi\mu^2H^2}{\rho_o}\right)\delta u_z = \rho_o\delta u_x\left\{a^2 n_z n_x\right\}+\rho_o\delta u_y\left\{a^2 n_z n_y + \frac{\mu H_x^2 n_z n_y}{\rho_o}\right\}+\rho_o\delta u_z\left\{a^2 n_z^2+\frac{\mu H_x^2}{\rho_o}(n_x^2+n_z^2)\right\}$$

We chose H to be in the x-direction. We lose no essential generality if we further choose $n_z = 0$. (Since the wave front normal can still be at an arbitrary angle to H.) Putting $n_z = 0$ in the above three equations, the determinant becomes

$$\begin{vmatrix} -\rho_o v^2\left(1+\frac{i_o}{c^2}\right)+\rho_o a^2 n_x^2 & \rho_o a^2 n_x n_y & 0 \\ \rho_o a^2 n_x n_y & -\rho_o v^2\left(1+\frac{i_o}{c^2}+\frac{\chi\mu^2H^2}{\rho_o}\right)+\rho_o a^2 n_y^2+\mu H^2 & 0 \\ 0 & 0 & -\rho_o v^2\left(1+\frac{i_o}{c^2}+\frac{\chi\mu^2H^2}{\rho_o}\right)+\mu H^2 n_x^2 \end{vmatrix}$$

Since H is in the x-direction we have $n_x = \frac{H_n}{|H|}$. Also, since $n_z = 0$,

we have $n_y^2 = 1 - n_x^2 = 1 - \frac{H_n^2}{H^2}$. Expressing the n_x and n_y in the above determinant in terms of H and H_n, it becomes the determinant given in $(7A_4)$.

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